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On the structure of the Figueroa unital and the existence of O'Nan configurations[☆]

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ABSTRACT

The finite Figueroa planes are non-Desarguesian projective planes of order q^3 for all prime powers $q > 2$, constructed algebraically in 1982 by Figueroa, and Hering and Schaeffer, and synthetically in 1986 by Grundhöfer. All Figueroa planes of finite square order are shown to possess a unitary polarity by de Resmini and Hamilton in 1998, and hence admit unitals. Hui and Wong (2012) have shown that these polar unitals do not satisfy a necessary condition, introduced by Wilbrink in 1983, for a unital to be classical, and hence they are not classical. In this article we introduce and make use of a new alternative synthetic description of the Figueroa plane and unital to demonstrate the existence of O'Nan configurations, thus providing support to Piper's conjecture (1981).

1. Introduction

A polar unital of order n, n a positive integer >2, is a unitary block design with parameters 2-$(n^3+1, n+1, 1)$ defined by a unitary polarity ρ of a projective plane π of order n^2: the points are the absolute points and the blocks are the non-absolute lines of ρ. If π is the classical plane $PG(2, q^2)$, then the polar unital is called a *classical unital*, and the point set is a *Hermitian curve*. Since all unitary polarities of $PG(2, q^2)$ are projectively equivalent, we refer to *the* classical unital, and denote it by $\mathcal{H}$. (For general reference see [10,11].)

A fundamental result concerning the classical unital is the determination of its design automorphism group. In 1972 O'Nan [15] proved that $\mathrm{Aut}(\mathcal{H})$, the design automorphism group, is isomorphic to $\mathrm{Col}(\mathcal{H})$, the collineation subgroup of $PG(2, q^2)$ stabilizing $\mathcal{H}$. It was also observed that in the classical unital there is no O'Nan configuration: four (non-absolute) lines intersecting in six (absolute) points. In 1981 Piper [16] conjectured that the non-existence of O'Nan configurations characterizes the classical unital. The conjecture remains open.

The Dickson–Ganley unital is another example of a polar unital [7] (1972). It is a unital given by a unitary polarity in the projective plane $\Pi(\mathcal{K})$ defined over a Dickson semifield $\mathcal{K}$. In [12] (2013), three different proofs are given to show that the Dickson–Ganley unital $\mathcal{U}(\sigma)$, parametrized by a field automorphism σ, is non-classical if σ is not the identity (if σ is the identity the unital is classical); one of the proofs is the demonstration of the existence of O'Nan configurations; and that the design automorphism group of $\mathcal{U}(\sigma)$, whether σ is the identity or not, is the collineation subgroup of the ambient plane stabilizing the unital. Thus the latter contains as a special case the corresponding result of O'Nan's mentioned above.

The Figueroa unital is also a polar unital. It is a unital of order q^3 defined by a unitary polarity in a Figueroa plane of order q^6, for q power of a prime. The Figueroa plane is a non-translation plane of order q^3, first constructed algebraically in 1982

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by Figueroa [6] (for some prime power orders $q > 2$) and Hering and Schaeffer [9] (for all prime power orders $q > 2$), and synthetically in 1986 by Grundhöfer [8]. The existence of a unitary polarity in any Figueroa plane of square order was discovered in 1998 by de Resmini and Hamilton [4], and the unital it defines has been referred to as a Figueroa unital [13].

The fact that the Figueroa unital is non-classical is a main result of [13]. The proof makes use of Wilbrink's characterization of the classical unital. In 1983 Wilbrink [17] gave three configurational requirements characterizing the classical unital, the first of which is the non-existence of O'Nan configurations. In [13] it is shown that the Figueroa unital does not satisfy the second condition of Wilbrink's. In view of Piper's conjecture, it is important to ask whether there exists an O'Nan configuration in the Figueroa unital (see Remark 4.6 of [13]). In this article we answer the question in the affirmative, thus providing support to Piper's conjecture, and at the same time another proof that the Figueroa unital is non-classical.

Below we give a brief description of each of the following sections.

In Section 2, we recall Grundhöfer's synthetic construction of the Figueroa plane. We then give a new alternative construction of the Figueroa plane, and explain why this is more suitable for our purpose.

In Section 3, we recall the Figueroa unital defined by the unitary polarity of de Resmini and Hamilton in the original construction of the Figueroa plane. We then reproduce the Figueroa unital with a new description under our alternative construction of the Figueroa plane. We explain why this is more suitable for our purpose, and outline a plan for the search of O'Nan configurations in the unital.

In Section 4, we carry out our plan for the search of an O'Nan configuration in the Figueroa unital.

2. An alternative description of the Figueroa plane

Let $PG(2, q^3)$ be the classical projective plane over the finite field $\mathbb{F}_{q^3}$, $q > 2$. This defines a symmetric BIBD $(\mathcal{P}, \mathcal{L}, \in)$ with parameters 2-$(q^6+q^3+1, q^3+1, 1)$. As usual a point in $\mathcal{P}$ is given by $[x_0, x_1, x_2]$ and a line in $\mathcal{L}$ by $[y_0, y_1, y_2]^t$. Denote by $p_1.p_2$ the unique line through the points p_1 and p_2, and $L_1.L_2$ the unique point on the lines L_1 and L_2.

Let α be a collineation of $PG(2, q^3)$ of order 3 fixing a subplane of order q pointwise (or equivalently, linewise). A point a is said to be of *type I* if $a^\alpha = a$, *type II* if the orbit of a under α are three collinear points, and *type III* if its orbit under α are three points in general position. Similarly we have the corresponding classification of lines into types. Denote by $\mathcal{P}_I$, $\mathcal{P}_{II}$ and $\mathcal{P}_{III}$ respectively the sets of points of types *I*, *II* and *III*. Similarly we have the partition of the lines $\mathcal{L}$ into the subsets $\mathcal{L}_I$, $\mathcal{L}_{II}$ and $\mathcal{L}_{III}$. By [13] (Lemma 2.1), on a type *I* line L there are $q+1$ type *I* points and q^3-q type *II* points; on a type *II* line L there are 1 type *I* point, q^2 type *II* points and q^3-q^2 type *III* points; on a type *III* line L there are q^2+q+1 type *II* points and q^3-q^2-q type *III* points. Moreover $|\mathcal{P}_I| = q^2+q+1$, $|\mathcal{P}_{II}| = (q^2+q+1)(q^3-q)$ and $|\mathcal{P}_{III}| = (q^3-q^2)(q^3-q)$. We have the dual statements for lines.

Following Grundhöfer [8] let μ be the involutory bijection between $\mathcal{P}_{III}$ and $\mathcal{L}_{III}$ given by $p^\mu = p^\alpha.p^{\alpha^2}$ for $p \in \mathcal{P}_{III}$, $L^\mu = L^\alpha.L^{\alpha^2}$ for $L \in \mathcal{L}_{III}$. The Figueroa plane, $\mathcal{F}_{q^3}$, is the symmetric BIBD $(\mathcal{P}, \mathcal{L}, \mathbf{I}_\mathcal{F})$ with parameters 2-$(q^6+q^3+1, q^3+1, 1)$ given as follows:

$$\begin{aligned} &p\,\mathbf{I}_\mathcal{F}\,L \Leftrightarrow L^\mu \in p^\mu \quad \text{if } p \in \mathcal{P}_{III}, L \in \mathcal{L}_{III};\\ &p\,\mathbf{I}_\mathcal{F}\,L \Leftrightarrow p \in L \quad \text{otherwise.}\end{aligned}$$

For our purpose we introduce the following alternative description of the Figueroa plane. Let ρ be a polarity in $PG(2, q^3)$ that commutes with α. For example, corresponding to the standard collineation $\alpha : [x_0, x_1, x_2] \mapsto [x_0^q, x_1^q, x_2^q]$, we can take the orthogonal polarity $\rho_1 : [x_0, x_1, x_2] \leftrightarrow [x_0, x_1, x_2]^t$; or, if q is a square, the unitary polarity $\rho_2 : [x_0, x_1, x_2] \leftrightarrow [x_0^{q^{3/2}}, x_1^{q^{3/2}}, x_2^{q^{3/2}}]$. Then ρ preserves types and commutes with μ. Let $\mathcal{F}_{\rho,q^3} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be the structure defined as follows:

$$\begin{aligned} &p\,\mathbf{I}\,L \Leftrightarrow p \in L^{\mu\rho} \quad \text{if } p \in \mathcal{P}_{II}, L \in \mathcal{L}_{III};\\ &p\,\mathbf{I}\,L \Leftrightarrow p^{\mu\rho} \in L \quad \text{if } p \in \mathcal{P}_{III}, L \in \mathcal{L}_{II};\\ &p\,\mathbf{I}\,L \Leftrightarrow p \in L \quad \text{otherwise.}\end{aligned}$$

Then $\mathcal{F}_{\rho,q^3}$ is an alternative description of the Figueroa plane, by the following:

Theorem 2.1. *$\mathcal{F}_{\rho,q^3}$ is isomorphic to $\mathcal{F}_{q^3}$.*

Proof. Let ϕ be the bijection from $\mathcal{P}$ to $\mathcal{P}$ and from $\mathcal{L}$ to $\mathcal{L}$ which sends a type *III* point p to (the type *III* point) $p^{\mu\rho}$, a type *III* line L to (the type *III* line) $L^{\mu\rho}$, and is the identity otherwise. Then ϕ defines an isomorphism between $\mathcal{F}_{\rho,q^3}$ and $\mathcal{F}_{q^3}$ provided $p\,\mathbf{I}\,L$ if and only if $p^\phi\,\mathbf{I}_\mathcal{F}\,L^\phi$. Since there is no incidence between a type *I* element and a type *III* element, we have only the following cases to verify:

If $p \in \mathcal{P}_{III}$ and $L \in \mathcal{L}_{III}$,

$$p\,\mathbf{I}\,L \Leftrightarrow p \in L \Leftrightarrow L^{\phi\mu} = L^{\mu\rho\mu} = L^{\rho\mu\mu} = L^\rho \in p^\rho = p^{\rho\mu\mu} = p^{\mu\rho\mu} = p^{\phi\mu} \Leftrightarrow p^\phi\,\mathbf{I}_\mathcal{F}\,L^\phi;$$

if $p \in \mathcal{P}_{III}$ and $L \in \mathcal{L}_{II}$,

$$p\,\mathbf{I}\,L \Leftrightarrow p^\phi = p^{\mu\rho} \in L = L^\phi \Leftrightarrow p^\phi\,\mathbf{I}_\mathcal{F}\,L^\phi;$$

if $p \in \mathcal{P}_{II}$ and $L \in \mathcal{L}_{III}$,

$$p\,\mathbf{I}\,L \Leftrightarrow p^{\phi} = p \in L^{\mu\rho} = L^{\phi} \Leftrightarrow p^{\phi}\,\mathbf{I}_{\mathcal{F}}\,L^{\phi};$$

if p and L are both not of type *III*,

$$p\,\mathbf{I}\,L \Leftrightarrow p \in L \Leftrightarrow p^{\phi} \in L^{\phi} \Leftrightarrow p^{\phi}\,\mathbf{I}_{\mathcal{F}}\,L^{\phi}.$$

Thus ϕ is indeed an isomorphism. □

In the Figueroa plane $\mathcal{F}_{\rho,q^3} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ we shall use the following notations: given two points p_1 and p_2 in $\mathcal{P}$, denote by $p_1 \cdot_{\mathbf{I}} p_2$ the unique line L in $\mathcal{L}$ such that $p_i\,\mathbf{I}\,L$ for $i = 1, 2$. Similarly, given two lines L_1 and L_2 in $\mathcal{L}$, denote by $L_1 \cdot_{\mathbf{I}} L_2$ the unique point p in $\mathcal{P}$ such that $p\,\mathbf{I}\,L_i$ for $i = 1, 2$.

It is important to note that Lemma 2.1 of [13] remains true for both $\mathcal{F}_{q^3}$ and $\mathcal{F}_{\rho,q^3}$. We illustrate the counting with two examples, one for $\mathcal{F}_{q^3}$ and another for $\mathcal{F}_{\rho,q^3}$. Given $L \in \mathcal{L}_{III}$, we first count the number of type *III* points on L in $\mathcal{F}_{q^3}$:

$$\begin{aligned} |\{(p, L) | p\,\mathbf{I}_{\mathcal{F}}\,L, p \in \mathcal{P}_{III}\}| &= |\{p \in \mathcal{P}_{III} | L^{\mu} \in p^{\mu}\}| \\ &= |\{M \in \mathcal{L}_{III} | L^{\mu} \in M\}| \\ &= q^3 - q^2 - q; \end{aligned}$$

next we count the number of type *II* points on L in $\mathcal{F}_{\rho,q^3}$:

$$\begin{aligned} |\{(p, L) | p\,\mathbf{I}\,L, p \in \mathcal{P}_{II}\}| &= |\{p \in \mathcal{P}_{II} | p \in L^{\mu\rho}\}| \\ &= q^2 + q + 1. \end{aligned}$$

This enables further counting of the following numbers of incident point-line pairs, or *flags*, (p, L):

$$\begin{aligned} |\{(p, L) | p\,\mathbf{I}_{\mathcal{F}}\,L, p \in \mathcal{P}_{III}, L \in \mathcal{L}_{III}\}| &= |\mathcal{L}_{III}|(q^3 - q^2 - q) \\ &= (q^3 - q^2)(q^3 - q)(q^3 - q^2 - q); \\ |\{(p, L) | p\,\mathbf{I}\,L, p \in \mathcal{P}_{II}, L \in \mathcal{L}_{III}\}| &= |\mathcal{L}_{III}|(q^2 + q + 1) \\ &= (q^3 - q^2)(q^3 - q)(q^2 + q + 1). \end{aligned}$$

Similarly,

$$|\{(p, L) | p\,\mathbf{I}\,L, p \in \mathcal{P}_{III}, L \in \mathcal{L}_{II}\}| = (q^3 - q^2)(q^3 - q)(q^2 + q + 1).$$

Since there is a total of $q^9 + 2q^6 + 2q^3 + 1$ flags, the number of flags which are cases described as "otherwise" in the definition of $\mathbf{I}_{\mathcal{F}}$ is $2q^8 + q^7 - q^6 + q^4 + 2q^3 + 1$, and the corresponding number for $\mathbf{I}$ is $q^9 - 2q^8 + 4q^6 + 2q^5 + 1$. Thus for large q, there are significantly more flags (p, L) defined by $p \in L$ for $\mathbf{I}$ than for $\mathbf{I}_{\mathcal{F}}$. This is an advantage of the alternative description $\mathcal{F}_{\rho,q^3}$ over $\mathcal{F}_{q^3}$. Furthermore, as we shall see in the next section, the alternative description is especially suitable for the study of the Figueroa unital.

Remark 2.2. The collineation group of the Figueroa plane has been studied by various authors (including Dempwolff [3] and completed by Batten and Johnson [2]). In [2] (Theorem 2 and a further remark), criteria are given for two Figueroa planes to be isomorphic (for both finite and infinite cases). Accordingly, in the finite case, different choices of α give rise to isomorphic Figueroa planes.

3. The Figueroa unital

From now on our main concern is with planes of square order. In the classical plane $PG(2, q^6)$ let α be the collineation of order 3 fixing a subplane of order q^2 pointwise, as defined in Section 2. Let ρ be a unitary polarity in $PG(2, q^6)$ which commutes with α. Denote by $\mathcal{A}$ the set of absolute points and $\mathcal{B}$ the set of non-absolute lines of ρ. The classical unital $\mathcal{H}$ is the unitary block design $(\mathcal{A}, \mathcal{B}, \in|_{\mathcal{A}\times\mathcal{B}})$ with parameters 2-$(q^9 + 1, q^3 + 1, 1)$. For simplicity in notation, we write the subdesign $\mathcal{H}$ of $(\mathcal{P}, \mathcal{L}, \in)$ as $(\mathcal{A}, \mathcal{B}, \in)$. We shall also adopt this convention for other subdesigns.

The set $\mathcal{A}$ is partitioned by types into subsets $\mathcal{A}_I = \mathcal{A} \cap \mathcal{P}_I$, $\mathcal{A}_{II} = \mathcal{A} \cap \mathcal{P}_{II}$ and $\mathcal{A}_{III} = \mathcal{A} \cap \mathcal{P}_{III}$. Similarly we have the partition of $\mathcal{B}$ into the subsets $\mathcal{B}_I$, $\mathcal{B}_{II}$ and $\mathcal{B}_{III}$. In 1998 de Resmini and Hamilton [4] proved that ρ is a unitary polarity of $\mathcal{F}_{q^6}$ and defines a unitary block design $\mathcal{U}_{\mathcal{F}} = (\mathcal{A}_I \cup \mathcal{A}_{II} \cup \mathcal{A}_{III}^{\mu\rho}, \mathcal{B}_I \cup \mathcal{B}_{II} \cup \mathcal{B}_{III}^{\mu\rho}, \mathbf{I}_{\mathcal{F}})$ with parameters 2-$(q^9 + 1, q^3 + 1, 1)$. In [13] this is called the Figueroa unital. Naturally one asks whether ρ remains a unitary polarity in $\mathcal{F}_{\rho,q^6}$. We show that this is the case, and that ρ defines a unitary block design isomorphic to $\mathcal{U}_{\mathcal{F}}$.

Theorem 3.1. *Let ρ be a unitary polarity in $PG(2, q^6)$ that commutes with α. Then ρ is a unitary polarity in $\mathcal{F}_{\rho,q^6}$, and defines a unitary block design $\mathcal{U}_{\mathcal{F}_\rho} = (\mathcal{A}, \mathcal{B}, \mathbf{I})$ with parameters 2-$(q^9 + 1, q^3 + 1, 1)$ which is isomorphic to the Figueroa unital $\mathcal{U}_{\mathcal{F}}$.*

Proof. To show that ρ is a polarity in $\mathcal{F}_{\rho,q^6}$ we only need to verify that $p \,\mathbf{I}\, L$ if and only if $L^\rho \,\mathbf{I}\, p^\rho$. There are two cases to check:

if $p \in \mathcal{P}_{II}$ and $L \in \mathcal{L}_{III}$,

$$p \,\mathbf{I}\, L \Leftrightarrow p \in L^{\mu\rho} \Leftrightarrow L^\mu \in p^\rho \Leftrightarrow L^{\rho\mu\rho} \in p^\rho \Leftrightarrow L^\rho \,\mathbf{I}\, p^\rho;$$

if $p \in \mathcal{P}_{III}$ and $L \in \mathcal{L}_{II}$,

$$p \,\mathbf{I}\, L \Leftrightarrow p^{\mu\rho} \in L \Leftrightarrow L^\rho \in p^\mu \Leftrightarrow L^\rho \in p^{\rho\mu\rho} \Leftrightarrow L^\rho \,\mathbf{I}\, p^\rho.$$

Since for elements of the same type $\mathbf{I}$ is the same as $\in$ and ρ preserves types, the set of absolute points and the set of non-absolute lines of ρ in $\mathcal{F}_{\rho,q^6}$ are respectively $\mathcal{A}$ and $\mathcal{B}$. It follows that ρ is a unitary polarity in $\mathcal{F}_{\rho,q^6}$ (there is the correct number of absolute points) and defines the unitary block design $\mathcal{U}_{\mathcal{F}_\rho} = (\mathcal{A}, \mathcal{B}, \mathbf{I})$ with parameters 2-$(q^9 + 1, q^3 + 1, 1)$.

Finally, since $\mathcal{F}_{\rho,q^6}$ and $\mathcal{F}_{q^6}$ are isomorphic via ϕ as given in the proof of Theorem 2.1, and since ρ commutes with ϕ, the unitary block designs $\mathcal{U}_{\mathcal{F}}$ and $\mathcal{U}_{\mathcal{F}_\rho}$ are isomorphic. $\square$

Note that the point sets of $\mathcal{H}$ and $\mathcal{U}_{\mathcal{F}}$ are different; the sets $\mathcal{A}_{III}$ and $\mathcal{A}_{III}^{\mu\rho}$ are disjoint by Lemma 3.1 of [13]. On the other hand, the point sets of $\mathcal{H}$ and $\mathcal{U}_{\mathcal{F}_\rho}$ are identical, and so are the block sets. Thus, when comparing the Figueroa unital with the classical unital, it is far more convenient to study $\mathcal{U}_{\mathcal{F}_\rho}$ rather than $\mathcal{U}_{\mathcal{F}}$. Indeed, using our description of the Figueroa unital, we shall demonstrate the existence of O'Nan configurations, and also extend the structural results in [13]. Since our focus will be on $\mathcal{U}_{\mathcal{F}_\rho}$ from now on, we denote it simply by $\mathcal{U}$.

We now outline the plan of our search for an O'Nan configuration in $\mathcal{U}$; the details of the proof will be given in Section 4. By Theorem 4.2 of [13], in both $\mathcal{A}$ and $\mathcal{B}$ the majority of the elements are of type *III*. It is natural to search for O'Nan configurations consisting only of type *III* elements. However, since $(\mathcal{A}_{III}, \mathcal{B}_{III}, \mathbf{I})$ and $(\mathcal{A}_{III}, \mathcal{B}_{III}, \in)$ are isomorphic incidence structures, there can be no such examples.

The same reasoning suggests that a probable plan is to search for an O'Nan configuration in $\mathcal{U}$ in which the four lines are all of type *III* and one of the six points is of type *II* with the rest of type *III*. Since incidence between points and lines of the same type is the same for $\mathbf{I}$ and $\in$, we are actually looking for four lines $N_1, N_2, N_3, N_4 \in \mathcal{B}_{III}$ such that $N_1.N_3$, $N_1.N_4$, $N_2.N_3$, $N_2.N_4$, $N_3.N_4 \in \mathcal{A}_{III}$, $N_1.N_2 \in \mathcal{P}_{II} \backslash \mathcal{A}_{II}$, but $N_1 ._{\mathbf{I}} N_2 \in \mathcal{A}_{II}$. We show that this can be done, as follows. Given certain configurations, we derive sufficient conditions for certain points to be absolute. This will involve the axioms of Pappus and Desargues. Since we require type *III* elements we obtain bounds for them in general as well as in specific situations. This will involve working with coordinates. With this preparation we proceed to construct two type *III* non-absolute lines meeting in a type *II* non-absolute point in the classical plane but an absolute point in the Figueroa plane. We then argue that this setup can be completed to an O'Nan configuration by adding two type *III* non-absolute lines so that in the resulting configuration the five remaining intersections are all absolute points of type *III*. An estimate is also given for the number of such O'Nan configurations.

4. The existence of O'Nan configurations

In this section we demonstrate the existence of O'Nan configurations in the Figueroa unital by carrying out the plan outlined in the end of Section 3.

We begin by proving that in a rather general situation there is a good supply of type *III* lines, and then deduce from it a lower bound for the number of type *III* points on a type *III* line.

In view of Remark 2.2, we may choose without loss of generality α as given by:

$$[x_0, x_1, x_2]^\alpha = [x_1^{q^2}, x_2^{q^2}, x_0^{q^2}] \quad \text{for } [x_0, x_1, x_2] \in \mathcal{P},$$

$$[y_0, y_1, y_2]^{t\alpha} = [y_1^{q^2}, y_2^{q^2}, y_0^{q^2}]^t \quad \text{for } [y_0, y_1, y_2]^t \in \mathcal{L}.$$

Then the type of a point $[x_0, x_1, x_2]$ is determined by the $\mathbb{F}_{q^6}$-rank of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1^{q^2} & x_2^{q^2} & x_0^{q^2} \\ x_2^{q^4} & x_0^{q^4} & x_1^{q^4} \end{bmatrix}.$$

If the rank is 1, then the point is of type *I*; if the rank is 2, then the point is of type *II*; if the rank is 3, then the point is of type *III*. Similarly, the type of a line $[y_0, y_1, y_2]^t$ is determined by the $\mathbb{F}_{q^6}$–rank of the matrix

$$\begin{bmatrix} y_0 & y_1^{q^2} & y_2^{q^4} \\ y_1 & y_2^{q^2} & y_0^{q^4} \\ y_2 & y_0^{q^2} & y_1^{q^4} \end{bmatrix}.$$

Theorem 4.1. *Let $p \in \mathcal{P}_{II} \cup \mathcal{P}_{III}$, $L \in \mathcal{B}$ such that $p \notin L$, and $L \cap \mathcal{A} = \{a_1, a_2, \ldots, a_{q^3+1}\}$. Then $\{p.a_i | i = 1, 2, \ldots, q^3 + 1\}$ contains at least $q^3 - q^2 - q$ type III lines.*

Proof. Let $p = [x_0, x_1, x_2]$. It is known that $L \cap \mathcal{A}$ is a Baer subline $PG(1, q^3)$ in $PG(2, q^6)$ (for example, see Lemmas 7.20 and 6.2 of [10]). Thus, there exists

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \in PGL(3, q^6)$$

such that $L = \{[0, 1, s]\mathbf{M} | s \in \mathbb{F}_{q^6}\} \cup \{[0, 0, 1]\mathbf{M}\}$, and $L \cap \mathcal{A} = \{a_1, a_2, \ldots, a_{q^3+1}\} = \{[0, 1, s]\mathbf{M} | s \in \mathbb{F}_{q^3}\} \cup \{[0, 0, 1]\mathbf{M}\}$. Then $\{p.x | x \in L\} = \{[f_1 + f_2 s, g_1 + g_2 s, h_1 + h_2 s]^t | s \in \mathbb{F}_{q^6}\} \cup \{[f_2, g_2, h_2]^t\}$, where $f_1 = x_1 m_{23} - x_2 m_{22}$, $f_2 = x_1 m_{33} - x_2 m_{32}$, $g_1 = x_2 m_{21} - x_0 m_{23}$, $g_2 = x_2 m_{31} - x_0 m_{33}$, $h_1 = x_0 m_{22} - x_1 m_{21}$, $h_2 = x_0 m_{32} - x_1 m_{31}$, and $\{p.a_i | i = 1, 2, \ldots, q^3 + 1\} = \{[f_1 + f_2 s, g_1 + g_2 s, h_1 + h_2 s]^t | s \in \mathbb{F}_{q^3}\} \cup \{[f_2, g_2, h_2]^t\}$. Now $[f_1 + f_2 s, g_1 + g_2 s, h_1 + h_2 s]^t$ is not of type *III* if and only if

$$\begin{vmatrix} f_1 + f_2 s & (h_1 + h_2 s)^{q^2} & (g_1 + g_2 s)^{q^4} \\ g_1 + g_2 s & (f_1 + f_2 s)^{q^2} & (h_1 + h_2 s)^{q^4} \\ h_1 + h_2 s & (g_1 + g_2 s)^{q^2} & (f_1 + f_2 s)^{q^4} \end{vmatrix} = 0,$$

i.e. if and only if the following polynomial vanishes:

$$n_1 s^{q^4+q^2+1} + n_2 s^{q^4+q^2} + n_3 s^{q^4+1} + n_4 s^{q^2+1} + n_5 s^{q^4} + n_6 s^{q^2} + n_7 s + n_8,$$

where the n_i's depend on the f_i's, g_i's and h_i's. These can be easily computed and their exact expressions are not important.

Since by assumption p is not of type *I*, $\{p.x | x \in L\}$ contains at least two type *III* lines. Therefore the polynomial is not the zero polynomial. When we restrict s to be in $\mathbb{F}_{q^3}$, this becomes the following polynomial of degree at most $q^2 + q + 1$ (and is not the zero polynomial):

$$n_1 s^{q^2+q+1} + n_2 s^{q^2+q} + n_4 s^{q^2+1} + n_3 s^{q+1} + n_6 s^{q^2} + n_5 s^{q} + n_7 s + n_8.$$

Hence, there are at most $q^2 + q + 1$ lines in $\{[f_1 + f_2 s, g_1 + g_2 s, h_1 + h_2 s]^t | s \in \mathbb{F}_{q^3}\}$ that are not of type *III*.

It is now clear that $\{p.a_i | i = 1, 2, \ldots, q^3 + 1\} = \{[f_1 + f_2 s, g_1 + g_2 s, h_1 + h_2 s]^t | s \in \mathbb{F}_{q^3}\} \cup \{[f_2, g_2, h_2]^t\}$ contains at most $q^2 + q + 2$ non-type *III* line. Since $q^2 + q + 2 < q^3 + 1$, at least one of the lines $p.a_i$, $i = 1, 2, \ldots, q^3 + 1$, is of type III. By the transitive action of the stabilizer group of a Baer subline in a projective line, we can repeat the above argument with another $\mathbf{M}$, with the further restriction that $[f_2, g_2, h_2]^t$ is a type *III* line. Then $\{p.a_i | i = 1, 2, \ldots, q^3 + 1\}$ contains at most $q^2 + q + 1$ non-type *III* lines, and thus contains at least $q^3 - q^2 - q$ type *III* lines. □

Corollary 4.2. *Given $L \in \mathcal{B}_{III}$, there are at least $q^3 - q^2 - q$ points in $\mathcal{A}_{III} \cap L$.*

Proof. Let $a_1, a_2, \ldots, a_{q^3+1}$ be the absolute points on L. Substitute $p = L^\rho$ in Theorem 4.1. Then among the lines $p.a_1, p.a_2, \ldots, p.a_{q^3+1}$, at least $q^3 - q^2 - q$ of them are of type *III*. But $a_i^\rho = L^\rho.a_i = p.a_i$, for $i = 1, 2, \ldots, q^3 + 1$. Since ρ preserves type, among the points $a_1, a_2, \ldots, a_{q^3+1}$, at least $q^3 - q^2 - q$ of them are of type *III*. □

Next we prepare Lemmas 4.3 and 4.4 in order to prove in Lemma 4.5 that we can find $N_1, N_2 \in \mathcal{B}_{III}$ with $N_1^\rho \notin N_2$ such that $N_1.N_2 \in \mathcal{P}_{II} \backslash \mathcal{A}_{II}$ but $N_1.N_2 \in \mathcal{A}_{II}$.

Let L be a type *II* line and $c = L.L^\alpha$ the unique type *I* point on L (see Fig. 1). Let p_1 be a type *III* point on L and consider the type *III* line p_1^μ. Let r be a type *II* point on p_1^μ and consider the unique type *I* line M on r meeting $p_1^{\mu\alpha}$ at r^α and $p_1^{\mu\alpha^2}$ at r^{α^2}. We are interested in finding a type *III* point $p(r) \neq p_1$ on L such that $r \in p(r)^\mu$.

Lemma 4.3. *In $PG(2, q^6)$, let L be a type II line and p_1 a type III point on L. Let $\mathcal{R} = \{r \in p_1^\mu | r \in \mathcal{P}_{II}, L.L^\alpha \notin r.r^\alpha, r \notin L, r \notin L^\alpha, r \notin L^{\alpha^2}\}$. Then the cardinality of $\mathcal{R}$ is at least $q^4 - 3$. Moreover, given any $r \in \mathcal{R}$, there exists $p(r) \in L \backslash \{p_1\}$, $p(r) \in \mathcal{P}_{III}$, such that $r \in p(r)^\mu$. Furthermore, $p(r) \neq p(\widehat{r})$ if $r \neq \widehat{r}$.*

Proof. We work in $PG(2, q^6)$. First, we estimate the cardinality of $\mathcal{R}$. Let $\mathcal{R}_1 = \{r \in \mathcal{P}_{II} | r \in p_1^\mu\}$, $\mathcal{R}_2 = \{r \in \mathcal{R}_1 | L.L^\alpha \in r.r^\alpha\}$ and $\mathcal{R}_3 = \{r \in \mathcal{R}_1 | r \in L \text{ or } r \in L^\alpha \text{ or } r \in L^{\alpha^2}\}$. Then $\mathcal{R} = \mathcal{R}_1 \backslash (\mathcal{R}_2 \cup \mathcal{R}_3)$. Now, the cardinality of $\mathcal{R}_1$ equals the number of type *II* points on a type *III* line, which is $q^4 + q^2 + 1$; the cardinality of $\mathcal{R}_2$ equals the number of type *I* lines on a type *I* point, which is $q^2 + 1$; the cardinality of $\mathcal{R}_3$ is at most 3. Thus, the cardinality of $\mathcal{R}$ is at least $q^4 - 3$.

For any $r \in \mathcal{R}$, let M be the type *I* line containing r, r^α and r^{α^2}. Let c be the type *I* point at which L, L^α and L^{α^2} are concurrent. Since $c \notin M$ is the only type *I* point on the type *II* line L, $L.M$ is of type *II*. So c, $L.M$ and p_1 are distinct points. Let $p_2, p_3, \ldots, p_{q^6-1}$ be the remaining points of L. For $i \in \{1, 2, 3, \ldots, q^6 - 1\}$, let $p_i' = (p_i.r^{\alpha^2}).L^\alpha$ and $p_i'' = (p_i.r^\alpha).L^{\alpha^2}$; then $\Delta_i = \{p_i, p_i', p_i''\}$ is a set of three non-collinear points (see Fig. 1). Note that $\Delta_1 = \{p_1, p_1^\alpha, p_1^{\alpha^2}\}$. Indeed, $r \in p_1^\mu = p_1^\alpha.p_1^{\alpha^2}$

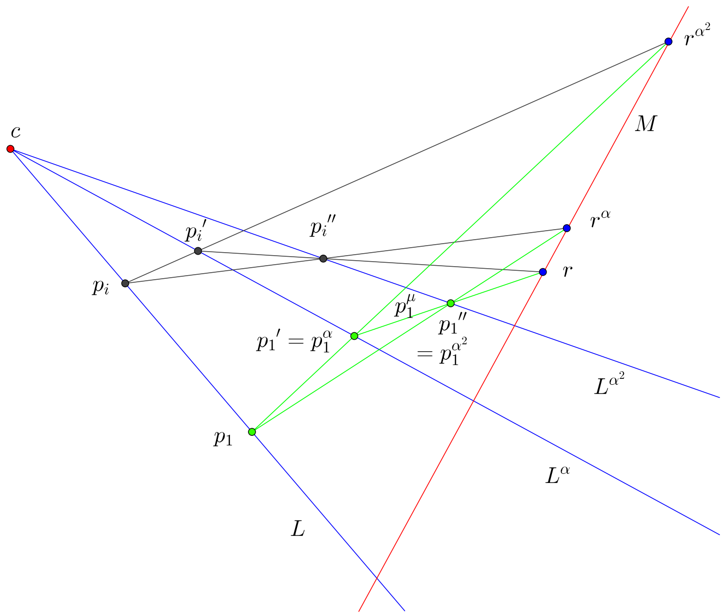

Fig. 1. Consequence of Desargues' Axiom.

implies that $r^\alpha \in p_1^{\alpha^2}.p_1$ and $r^{\alpha^2} \in p_1.p_1^\alpha$; it follows that $p_1' = (p_1.r^{\alpha^2}).L^\alpha = (p_1.p_1^\alpha).L^\alpha = p_1^\alpha$ and $p_1'' = (p_1.r^\alpha).L^{\alpha^2} = (p_1^{\alpha^2}.p_1).L^{\alpha^2} = p_1^{\alpha^2}$. We study Δ_i^α. We claim that for each i there exists a unique j such that $\Delta_i^\alpha = \Delta_j$, $1 \le i, j \le q^6 - 1$. This is a consequence of Desargues' Axiom, as follows: for any $i \in \{2, 3, \dots, q^6 - 1\}$, $p_1.p_i = L$, $p_1^\alpha.p_i' = L_1^\alpha$ and $p_1^{\alpha^2}.p_i'' = L_1^{\alpha^2}$ are concurrent at c. By Desargues' Axiom, $(p_i.p_i').(p_1.p_1^\alpha) = r^\alpha$, $(p_i.p_i'').(p_1.p_1^{\alpha^2}) = r^{\alpha^2}$ and $(p_i'.p_i'').(p_1^\alpha.p_1^{\alpha^2})$ are collinear on $r^\alpha.r^{\alpha^2} = M$. So $(p_i'.p_i'').(p_1^\alpha.p_1^{\alpha^2}) = r$. Let $p_j = p_i''^\alpha$. Then $\Delta_i^\alpha = \Delta_j$, as we wished. (Indeed, p_i, p_i'', r^α are collinear so $p_i^\alpha \in p_i''^\alpha.r^{\alpha^2}$, and since $p_i^\alpha \in L^\alpha$, $p_i^\alpha = (p_i''^\alpha.r^{\alpha^2}).L^\alpha = p_j'$; similarly, using the fact that p_i', p_i'', r are collinear we deduce that $p_i'^\alpha = p_j''$.) Now $\{\Delta_i | i = 1, 2, 3, \dots, q^6 - 1\}$ is partitioned under α into orbits of lengths either 1 or 3; in particular we have shown above that the orbit of Δ_1 is Δ_1 itself. Since $q^6 - 2$ is not divisible by 3, there exists $i \in \{2, 3, \dots, q^6 - 1\}$, say 2, such that $\Delta_2^\alpha = \Delta_2$. Then $\{p_2, p_2', p_2''\} = \{p_2, p_2^\alpha, p_2^{\alpha^2}\}$, and so p_2 is a type *III* point and $r \in p_2'.p_2'' = p_2^\alpha.p_2^{\alpha^2} = p_2^\mu$. Take $p(r) = p_2$.

If $\widehat{r}$ is different from r, then $p(\widehat{r}) \neq p(r)$. Otherwise $p(\widehat{r})' = p(r)'$; but then $p(\widehat{r})' = (p(\widehat{r}).\widehat{r}^{\alpha^2}).L^\alpha = (p(r).\widehat{r}^{\alpha^2}).L^\alpha$, whereas $p(r)' = (p(r).r^{\alpha^2}).L^\alpha$, and the two are not equal. □

We also have the dual version of Lemma 4.3:

Lemma 4.4. *In $PG(2, q^6)$, suppose $L_1 \in \mathcal{L}_{III}$ and $a \in \mathcal{P}_{II}$ such that $a \in L_1$. Let $\mathcal{M} = \{M \in \mathcal{L}_{II} | L_1^\mu \in M, M.M^\alpha \notin a.a^\alpha, a \notin M, a^\alpha \notin M, a^{\alpha^2} \notin M\}$. Then the cardinality of $\mathcal{M}$ is at least $q^4 - 3$. Moreover, given any $M \in \mathcal{M}$, there exists $L_2 \in \mathcal{L}_{III}$ such that $L_2 \neq L_1$, $a \in L_2$ and $L_2^\mu \in M$. Furthermore, different choices of M give different corresponding L_2.*

Using Corollary 4.2 and Lemma 4.4, we prove the following key lemma:

Lemma 4.5. *Given any $a \in \mathcal{A}_{II}$, there exist two lines $N_1, N_2 \in \mathcal{B}_{III}$ such that $N_1^\rho \notin N_2$, $N_1.N_2 \in \mathcal{P}_{II} \setminus \mathcal{A}_{II}$, and $N_1 ._{\mathbf{I}} N_2 = a$.*

Proof. We work in $PG(2, q^6)$ unless specified otherwise. Pick any $L_1 \in \mathcal{B}_{III}$ such that $a \in L_1$. Then L_1^μ is non-absolute; otherwise the six absolute points $L_1^\mu, L_1^{\mu\alpha}, L_1^{\mu\alpha^2}, a, a^\alpha, a^{\alpha^2}$ and the four non-absolute lines $L_1, L_1^\alpha, L_1^{\alpha^2}, a.a^\alpha$ constitute an O'Nan configuration of $\mathcal{H}$, a contradiction. Thus the line $N_1 = L_1^{\mu\rho}$ lies in $\mathcal{B}_{III}$ and contains a in $\mathcal{U}$. We proceed to find N_2. By Corollary 4.2, there are at most $q^2 + q + 1$ type *II* absolute points on N_1. Since ρ preserves types, there are at most $q^2 + q + 1$ type *II* absolute lines on $N_1^\rho = L_1^\mu$. Let $\mathcal{M} = \{M \in \mathcal{L}_{II} | L_1^\mu \in M, M.M^\alpha \notin a.a^\alpha, a \notin M, a^\alpha \notin M, a^{\alpha^2} \notin M\}$. By Lemma 4.4, the cardinality of $\mathcal{M}$ is at least $q^4 - 3$, which is greater than $q^2 + q + 1 + 2$, and so there are at least two non-absolute lines M, M' in $\mathcal{M}$. Again by Lemma 4.4 there exist two distinct type *III* lines L_2, L_2' on a, both different from L_1, such that $L_2^\mu \in M$ and $L_2'^\mu \in M'$. We claim that $L_2^\mu.L_2'^\mu$ is a type *II* line. This is a consequence of Desargues' Axiom, as follows. The three points $L_2.L_2' = a$, $L_2^\alpha.L_2'^\alpha = a^\alpha$ and $L_2^{\alpha^2}.L_2'^{\alpha^2} = a^{\alpha^2}$ are collinear, since a is of type *II*. By Desargues' Axiom, the three lines $L_2^\mu.L_2'^\mu$, $L_2^{\alpha\mu}.L_2'^{\alpha\mu} = (L_2^\mu.L_2'^\mu)^\alpha$ and $L_2^{\alpha^2\mu}.L_2'^{\alpha^2\mu} = (L_2^\mu.L_2'^\mu)^{\alpha^2}$ are concurrent. Thus, $L_2^\mu.L_2'^\mu$ is a type *II* line. Since the lines are of different types, $L_2^\mu.L_2'^\mu \neq N_1$. Either $L_2^\mu \notin N_1$ or $L_2'^\mu \notin N_1$ or both. Let us say $L_2^\mu \notin N_1$. Let $N_2 = L_2^{\mu\rho}$ and $x = N_1.N_2$. Then $N_1^\rho \notin N_2$ since

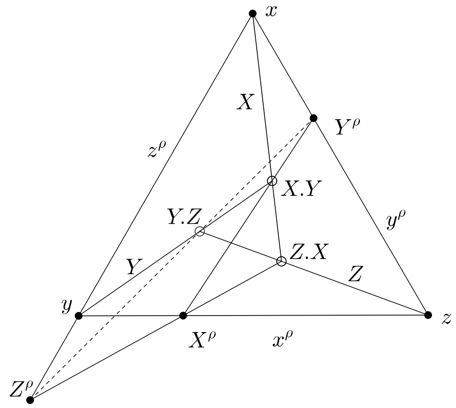

Fig. 2. Consequence of Pappus' Axiom.

$L_2^\mu \notin N_1$. Note that $a \in L_1 = N_1^{\mu\rho}$ and $a \in L_2 = N_2^{\rho\mu}$. By definition of $\mathbf{I}$, $a\,\mathbf{I}\,N_1$, $a\,\mathbf{I}\,N_2$, and thus $N_1 ._{\mathbf{I}} N_2 = a$. We also verify that $x = N_1.N_2 = L_1^{\mu\rho}.L_2^{\mu\rho} = (L_1^\mu.L_2^\mu)^\rho = M^\rho \in \mathcal{P}_{II} \setminus \mathcal{A}_{II}$. Finally, $N_2 \in \mathcal{B}_{III}$ since the type *III* line N_2 contains a type *II* absolute point a in $\mathcal{U}$. □

In order to complete the configuration obtained in Lemma 4.5 to an O'Nan configuration, we prepare three lemmas which let us determine when certain points in certain configurations must be absolute. These results are consequences of the axioms of Pappus and Desargues and are applicable to any polarity in any classical plane. In case of a unitary polarity, Lemmas 4.6 and 4.7 are equivalent to the three ruling families in a classical unital studied by Baker et al. [1] and Dover [5].

Lemma 4.6. *Let ρ be a polarity in a classical projective plane and (x, X), (y, Y), (z, Z) three flags of non-absolute point-line pairs with respect to ρ. Suppose $x = y^\rho.z^\rho$, $y = z^\rho.x^\rho$ and $z = x^\rho.y^\rho$. If both $Z.X$ and $X.Y$ are absolute, then $Y.Z$ is absolute.*

Proof. We skip the trivial case when X, Y, Z are concurrent. Then the line X contains the three points x, $Z.X$ and $X.Y$; while the line x^ρ contains the three points X^ρ, y and z (see Fig. 2). By Pappus' Axiom, the following three points are collinear;

$$\begin{aligned} ((Z.X).X^\rho).(x.y) &= (Z.X)^\rho.z^\rho && \text{(since } Z.X \text{ is absolute)} \\ &= ((Z.X).z)^\rho && \text{(since } \rho \text{ is a polarity)} \\ &= Z^\rho && \text{(since } z \text{ is on } Z), \end{aligned}$$

$$\begin{aligned} (x.z).((X.Y).X^\rho) &= y^\rho.(X.Y)^\rho && \text{(since } X.Y \text{ is absolute)} \\ &= (y.(X.Y))^\rho && \text{(since } \rho \text{ is a polarity)} \\ &= Y^\rho && \text{(since } y \text{ is on } Y), \end{aligned}$$

$$((X.Y).y).((Z.X).z) = Y.Z \qquad \text{(since } X.Y, y \in Y \text{ and } Z.X, z \in Z).$$

Now $Y.Z \in Y^\rho.Z^\rho = (Y.Z)^\rho$ implies $Y.Z$ is absolute. □

Lemma 4.7. *Suppose x and y are two non-absolute points of a polarity ρ in a classical projective plane such that $x \in y^\rho$, or equivalently $y \in x^\rho$. Let X_1, X_2 be two non-absolute lines through x, and Y_1, Y_2 two non-absolute lines through y. If the three points $X_1.Y_1$, $X_1.Y_2$, $X_2.Y_1$ are absolute, then $X_2.Y_2$ is absolute.*

Proof. Let $z = x^\rho.y^\rho$. Then z is non-absolute; otherwise $z \in z^\rho = x.y$ so that $x \in y.z = x^\rho$ and x is absolute, a contradiction. It follows that x, y, z are in general position and satisfy the hypothesis of Lemma 4.6. Now let $Z = z.(X_1.Y_1)$ (see Fig. 3). We apply Lemma 4.6 repeatedly, as follows. Firstly, $X_2.Y_1$ and $Y_1.Z = X_1.Y_1$ are both absolute, so $Z.X_2$ is absolute. Secondly, $X_1.Y_2$ and $X_1.Z = X_1.Y_1$ are both absolute, so $Z.Y_2$ is absolute. Finally, $X_2.Z$ and $Z.Y_2$ are both absolute, so $X_2.Y_2$ is absolute. □

In case ρ is a unitary polarity, the version of Lemma 4.7, in which the hypothesis "that x is a non-absolute point" is replaced by "that x is an absolute point", is Lemma A.3 of [14] and is a consequence of Desargues' Axiom. Using this result, we deduce the following:

Lemma 4.8. *Let ρ be a unitary polarity in a finite classical projective plane, a an absolute point, and B_1, B_2 two non-absolute lines not incident with a. If there exist three absolute points a_1, a_2, a_3 on B_1 such that $a_1' = (a.a_1).B_2$, $a_2' = (a.a_2).B_2$ and $a_3' = (a.a_3).B_2$ are absolute points, then $B_1.B_2$ is on a^ρ.*

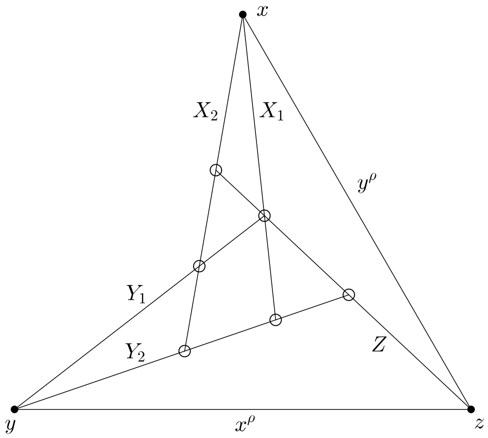

Fig. 3. Consequence of Lemma 4.6.

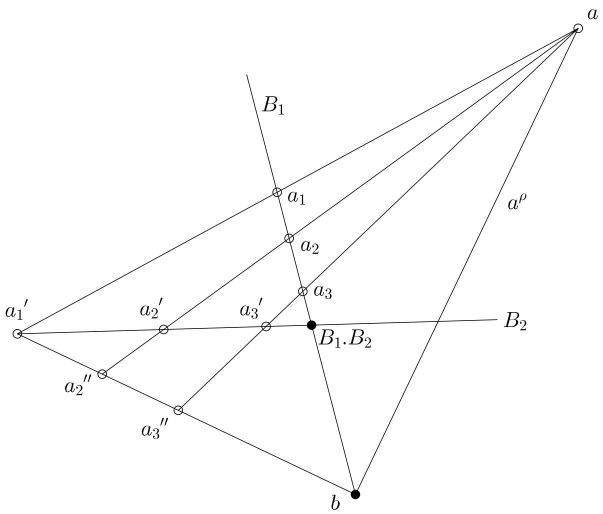

Fig. 4. Proof of Lemma 4.8 by contradiction.

Proof. We prove by contradiction (see Fig. 4). Assume to the contrary that $B_1.B_2$ is not on a^{ρ}. Denote $b = a^{\rho}.B_1 \neq B_1.B_2$. Since $b \in a^{\rho}$ and a_1, a_2, a_1' are absolute, $a_2'' = (b.a_1').(a.a_2)$ is absolute by Lemma A.3 of [14]. Similarly, $a_3'' = (b.a_1').(a.a_3)$ is absolute. Then the absolute points $a, a_1', a_2', a_2'', a_3'$ and a_3'' and the non-absolute lines $a.a_2, a.a_3, B_2$ and $b.a_1'$ constitute an O'Nan configuration, which is a contradiction. □

We are ready to prove the main theorem in this section.

Theorem 4.9. *There exists an O'Nan configuration in the Figueroa unital $\mathcal{U}$ of order q^3.*

Proof. We require $q^3 - 4q^2 - 4q - 5 > 0$. For $q = 2, 3, 4, 5$ we verify the existence of O'Nan configurations in such cases with the aid of a computer (see Table 1). We assume $q > 7$ from now on.

We begin with the set-up given by Lemma 4.5, i.e. $a \in \mathcal{A}_{II}, x \in \mathcal{P}_{II} \backslash \mathcal{A}_{II}$ and $N_1, N_2 \in \mathcal{B}_{III}$ such that $N_1.N_2 = a, N_1.N_2 = x$ and $N_1^{\rho} \notin N_2$. We proceed to find two lines $N_3, N_4 \in \mathcal{B}_{III}$, such that N_1, N_2, N_3, N_4 are in general position in $PG(2, q^6)$ and $N_1.N_3, N_1.N_4, N_2.N_3, N_2.N_4, N_3.N_4 \in \mathcal{A}_{III}$. By the definition of $\mathcal{F}_{\rho,q^6}$ and Theorem 3.1, the four lines N_1, N_2, N_3 and N_4 and their intersections in the Figueroa unital constitute an O'Nan configuration in the Figueroa unital.

First we find N_3. As in Lemma 4.5 we work in $PG(2, q^6)$ unless specified otherwise. Let $y_1 = N_1.x^{\rho}$ and $y_2 = N_2.x^{\rho}$. Since $N_1 = x.y_1$ is non-absolute, x, y_1 and N_1^{ρ} are three non-collinear points. Thus $y_1 \notin x.N_1^{\rho} = y_1^{\rho}$ and so y_1 is non-absolute. Similarly y_2 is non-absolute. Note that $N_1^{\rho} \in x^{\rho} \backslash \{y_1, y_2\}$ since N_1 is non-absolute and $N_1^{\rho} \notin N_2$.

Let a' be any point of $N_1 \cap \mathcal{A}$. Then $N_1^{\rho} = x^{\rho}.a'^{\rho}$, and $y_2 \notin a'^{\rho}$. Let $N_2 \cap \mathcal{A} = \{a_1'', \ldots, a_{q^3+1}''\}$ and $\mathcal{N} = \{a'.a_i'' | i = 1, \ldots, q^3 + 1\}$. Let $x_i' = (a'.a_i'').x^{\rho}, i = 1, \ldots, q^3 + 1$. By the contrapositive of Lemma 4.8 (applied to the absolute point a' and the two non-absolute lines N_2 and x^{ρ}), among these $q^3 + 1$ points there are at most two absolute points and hence at

Table 1
Examples of O'Nan configurations in small order Figueroa planes. The plane is generated from the Desarguesian plane over $\mathbb{F}_{q^6}$, where ω is a root of the given primitive polynomial over the prime field. We use the standard collineation $\alpha : [x_0, x_1, x_2] \mapsto [x_0^{q^2}, x_1^{q^2}, x_2^{q^2}]$ and the standard unitary polarity $\rho : [x_0, x_1, x_2] \leftrightarrow [x_0^{q^3}, x_1^{q^3}, x_2^{q^3}]^t$ to generate the Figueroa plane $\mathcal{F}_{\rho,q^6}$ and the Figueroa unital $\mathcal{U}$.

	$q = 2$	$q = 3$
Primitive polynomial	$x^6 + x^5 + x^4 + x + 1$	$x^6+2x^5+2x^4+2x^3+2x^2+2x+2$
Lines:		
N_1	$[1, \omega^{13}, \omega^{26}]^t$	$[1, \omega^{40}, \omega^{468}]^t$
N_2	$[1, \omega^{19}, \omega^{47}]^t$	$[1, \omega^{606}, \omega^{468}]^t$
N_3	$[1, \omega^{22}, \omega^{17}]^t$	$[1, \omega^{601}, \omega^{551}]^t$
N_4	$[1, \omega^{27}, \omega^{41}]^t$	$[1, \omega^{710}, \omega^{523}]^t$
Points:		
$N_1 ._I N_2 = N_1^{\mu\rho}.N_2^{\mu\rho}$	$[0, 1, \omega^7]$	$[0, 1, \omega^{13}]$
$N_1 ._I N_3 = N_1.N_3$	$[1, \omega^5, \omega]$	$[1, \omega^2, \omega^{566}]$
$N_1 ._I N_4 = N_1.N_4$	$[1, \omega^{62}, \omega^{18}]$	$[1, \omega^{308}, \omega^{515}]$
$N_2 ._I N_3 = N_2.N_3$	$[1, \omega^{11}, \omega^{20}]$	$[1, \omega^9, \omega^{461}]$
$N_2 ._I N_4 = N_2.N_4$	$[1, \omega^{32}, \omega^{48}]$	$[1, \omega^{77}, \omega^{342}]$
$N_3 ._I N_4 = N_3.N_4$	$[1, \omega^{30}, \omega^{38}]$	$[1, \omega^{589}, \omega^{452}]$
	$q = 4$	$q = 5$
Primitive polynomial	$x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^3+1$	$x^6+4x^5+4x^4+4x^3+4x^2+4x+2$
Lines:		
N_1	$[1, \omega^{692}, \omega^{1301}]^t$	$[1, \omega^{3519}, \omega^{3475}]^t$
N_2	$[1, \omega^{1656}, \omega^{415}]^t$	$[1, \omega^{5954}, \omega^{9820}]^t$
N_3	$[1, \omega^{171}, \omega^{3202}]^t$	$[1, \omega^{1537}, \omega^{3215}]^t$
N_4	$[1, \omega^{3199}, \omega^{3159}]^t$	$[1, \omega^{705}, \omega^{7064}]^t$
Points:		
$N_1 ._I N_2 = N_1^{\mu\rho}.N_2^{\mu\rho}$	$[0, 1, \omega^{63}]$	$[0, 1, \omega^{62}]$
$N_1 ._I N_3 = N_1.N_3$	$[1, \omega^{95}, \omega^{92}]$	$[1, \omega^{24}, \omega^{11180}]$
$N_1 ._I N_4 = N_1.N_4$	$[1, \omega^{2216}, \omega^{1057}]$	$[1, \omega^{9084}, \omega^{11104}]$
$N_2 ._I N_3 = N_2.N_3$	$[1, \omega^{29}, \omega^{1229}]$	$[1, \omega^{93}, \omega^{4836}]$
$N_2 ._I N_4 = N_2.N_4$	$[1, \omega^{183}, \omega^{2519}]$	$[1, \omega^{348}, \omega^{15620}]$
$N_3 ._I N_4 = N_3.N_4$	$[1, \omega^{2786}, \omega^{2986}]$	$[1, \omega^{4041}, \omega^{4973}]$

least $q^3 - 1$ non-absolute points. Let x' be one of these $q^3 - 1$ non-absolute points such that both x' and $x'' = x^\rho.x'^\rho$ are not of type *I*. Since there is only one type *I* point on a type *II* line and x^ρ is of type *II*, there are at least $q^3 - 3$ choices for x'.

Re-indexing if necessary we may assume that $x' = x_1'$. Consider the family of non-absolute lines $\mathcal{N}' = \{N_i' = x_1'.a_i'' | i = 1, 2, \dots, q^3 + 1\}$ (N_1' is non-absolute implies $x_1'^\rho \neq N_2$ and hence N_i' is non-absolute for any i). By Lemma 4.7 applied to the points x and x_1' and the lines N_1, N_2, N_1' and N_i' ($i \neq 1$), we have $N_i'.N_1 \in \mathcal{A}$, for $i = 2, \dots, q^3 + 1$. By Theorem 4.1 applied to the point x_1' and the line N_2, there are at least $q^3 - q^2 - q$ type *III* lines in $\mathcal{N}'$. Applying 4.2 to N_1 and then to N_2, there are at least $q^3 - q^2 - q - 2(q^2 + q + 1) = q^3 - 3q^2 - 3q - 2$ type *III* lines in $\mathcal{N}'$ each meeting N_1 and N_2 respectively at type *III* (absolute) points. Let N_3 be one of these lines.

Next we find N_4. We show that the non-type *I* point x'' is a non-absolute point distinct from both y_1 and y_2. Since x' is non-absolute, x, x' and x'' are three non-collinear points and hence $x'' \notin x''^\rho = x.x'$ is non-absolute. Since $N_3 = x'.a_i''$, for some i, is non-absolute, $N_1 \neq x'^\rho$ and hence $x'' \neq y_1$. Similarly, $x'' \neq y_2$. (We have omitted subscripts irrelevant to the argument.) Let $\mathcal{N}'' = \{N_i'' = x''.a_i'' | i = 1, 2, \dots, q^3 + 1\}$. For $i = 1, 2, \dots, q^3 + 1$, we have $N_i''.N_2, N_2.N_3 \in \mathcal{A}$. By Lemma 4.6 applied to the flags (x, N_2) (x', N_3) and (x'', N_i''), we have $N_i''.N_3 \in \mathcal{A}$. Then we apply Lemma 4.6 again to the flags (x', N_3), (x'', N_i'') and (x, N_1) and conclude that $N_1.N_i'' \in \mathcal{A}$. In other words, a line through x'' meets N_1 at an absolute point if and only if it meets N_2 at an absolute point, and also if and only if it meets N_3 at an absolute point. (Since ρ is a unitary polarity, the use of Lemmas 4.6 and 4.7 is equivalent to the use of the three ruling families studied in Baker et al. [1] and Dover [5].) Let $\mathcal{N}''' = \mathcal{N}'' \backslash \{x''.(N_1.N_3), x''.(N_2.N_3)\}$ so that any line in $\mathcal{N}'''$ is in general position with N_1, N_2 and N_3. Counting as before, using Theorem 4.1 applied to the point x'' and the line N_2, and Corollary 4.2 applied three times to N_1, N_2 and N_3, we conclude that there are at least $q^3 - 1 - 4(q^2 + q + 1) = q^3 - 4q^2 - 4q - 5$ type *III* lines in $\mathcal{N}'''$ each meeting N_1, N_2 and N_3 respectively at type *III* (absolute) points. Let N_4 be one of these lines. We have achieved our goal. □

Remark 4.10. Following the proof of Theorem 4.9, one counts at least $(q^3 - 3)(q^3 - 3q^2 - 3q - 2)(q^3 - 4q^2 - 4q - 5)$ ways to complete the setup in Lemma 4.5 to an O'Nan configuration in a Figueroa unital of order $q \geq 7$.

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